

On thermodynamic states of the Ising model on scale-free graphs

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Abstract

There is proposed a model of scale-free random graphs which are locally close to the uncorrelated complex random networks with divergent $\langle k^2 \rangle$ studied in e.g. S. N. Dorogovtsev *et al*, Rev. Mod. Phys. **80**, 1275 (2008). It is shown that the Ising model on these graphs with interaction intensities of arbitrary signs with probability one is in a paramagnetic state at sufficiently high finite values of the temperature. For the same graphs, the bond percolation model with probability one is in a nonpercolative state for positive values of the percolation probability. Possible extensions are discussed.

1 Introduction

In statistical physics, a paradigm model for cooperative phenomena is the Ising model of interacting spins attached to the vertices of a graph $G = (V, E)$. This model is described by the Hamiltonian

$$H = - \sum_{\{x,y\} \in E} J_{xy} \sigma_x \sigma_y - h \sum_{x \in V} \sigma_x, \quad (1.1)$$

where J_{xy} are interaction intensities, h is an external field, and the spins σ_x take values ± 1 . The sums run over the sets of edges E and vertices V of the graph G , which is supposed to be connected and countably infinite. In view of the latter fact, the Hamiltonian (1.1) has no direct mathematical meaning and is used as a formula for local Hamiltonians, H_Λ , which one obtains by restricting the sums to finite subsets $\Lambda \subset V$ and $E_\Lambda \subset E$, where

E_Λ is the collection of edges with both ends in Λ . Then the thermodynamic properties of (1.1) are studied in the limit $\Lambda \rightarrow V$. In a more sophisticated theory [1], a thermodynamic state is a Gibbs measure, and a phase transition is associated with the possibility for multiple thermodynamic states to exist at the same values of the temperature and h .

Obviously, the properties of the model (1.1) are closely related to those of the underlying graph. The case of a special interest is where this graph is random, with vertices having arbitrary number of neighbors. Then the graph itself can be characterized by the properties of the corresponding Ising model. Perhaps in view of this fact, methods of statistical physics are widely used also in the study of the underlying random graphs. A long list of publications on this topic can be found in the review articles [2, 3, 4].

For a vertex x , the degree $k(x)$ is defined as the number of neighbors of x , i.e. the number of $y \in V$ such that $\{x, y\} \in E$. If G is random, the degree of each $x \in V$ is an integer valued random variable and hence is characterized by the probability distribution

$$\text{Prob}\{k(x) = k\} = p_k(x), \quad k \geq 1, \quad x \in V. \quad (1.2)$$

In simple cases, the degrees $k(x)$ are independent and identically distributed, which in particular means that the probabilities $p_k(x)$ are the same for all x . If $p_k = Ck^{-\lambda}$, at least for big enough k and an appropriate $C > 0$, the corresponding graph is said to be scale-free, c.f. [5]. In this case, the m -th moment

$$\langle k^m \rangle = \sum_{k \geq 1} k^m p_k, \quad m \geq 1,$$

exists provided $\lambda > m + 1$. Notably, $\langle k^2 \rangle = +\infty$ for many real-world networks [4].

An example of a random graph with independent $k(x)$ is a Galton-Watson tree, which is the genealogic tree of a Galton-Watson branching process [6, 7]. In more complex models, the graph G is obtained, more or less explicitly, as the limit of an increasing sequence of finite random graphs, (V_n, E_n) , $n \in \mathbb{N}$, see, e.g. [8, 9, 10, 11]. In the physical literature, such constructions are rather informal that leaves certain possibilities for different interpretations, which we are going to use in this work. In 2002, almost simultaneously there appeared two papers [12, 13] where the model (1.1) on a scale-free graph was studied. The starting point is a scale-free distribution $\{p_k\}_{k \in \mathbb{N}}$, which “characterizes the entire net”, whereas the “local structure of connections” is characterized by “the distribution of the number of connections of the nearest neighbor of a vertex”, which is taken to be $kp_k/\langle k \rangle$, see the beginning of the first section in [12]. In [13], a similar assumption was made, see equation (3) thereof, and also (1) in [3]. In the latter article, such graphs are called *complex networks* in contrast to the classical models called *simple networks*, one of which is presumably the Galton-Watson tree

mentioned above. In both papers [12, 13], it was found that the model (1.1) with $h = 0$, ferromagnetic interactions $J_{xy} = J > 0$, and $\langle k^2 \rangle = +\infty$ is in an ordered (ferromagnetic) state at all temperatures¹ $T > 0$. Recently, in [10, 11] the mathematical construction of a random graph model was presented, in which the mentioned properties were realized. For this model, the conclusion of [12, 13] can be mathematically proven, see [15] and Subsection 5.2 below. The aim of this work is to construct “uncorrelated complex networks” with $\langle k^2 \rangle = \infty$, consistent with the assumptions made in [3, 12, 13], for which the model (1.1) with interaction intensities of arbitrary signs can be in a paramagnetic state at finite temperatures. In Section 2, we perform this construction and obtain a graph model, which is a combination of the one used in [10, 11] and of the ordinary Galton-Watson tree. In Section 3, we prove Theorem 3.1 the main result of which is that the model (1.1) on the introduced graphs, with any signs of J_{xy} , can be in a paramagnetic state at finite temperatures and $h = 0$, provided $\langle k^{1+\alpha} \rangle < \infty$ for some $\alpha > 0$. This covers the case of scale-free distributions with $\lambda \in (2, 3)$. The proof of Theorem 3.1 relies upon two lemmas, proven in Section 4. The result of Theorem 3.1 and the graph model introduced in Section 2 are discussed in the concluding Section 5, where we give also an extension of the results of [12, 13] concerning the model (1.1) with interaction intensities of arbitrary signs. Except possibly for Section 4, the paper is reasonably self-contained and accessible to non-mathematicians.

2 The graph models

Since the Galton-Watson tree will serve us as a prototype, we begin by presenting this model. More on this topic can be found in [6, 7, 16].

2.1 The Galton-Watson tree

To exclude the possibility of obtaining a finite tree, or a tree of bounded degree, we shall suppose that the basic random variable X takes positive integer values $k \in \mathbb{N}$ with probability $\text{Prob}(X = k) = p_{k+1}$ such that

$$p_k \in [0, 1), \quad \sum_{k \geq 2} p_k = 1, \quad \sum_{k=2}^n p_k < 1, \quad \text{for any } n \in \mathbb{N}, \quad (2.1)$$

$$a := \sum_{k \geq 2} k p_k < \infty.$$

One observes that

$$a > 2, \quad (2.2)$$

¹In contrast to regular graphs, e.g. crystal lattices, the Ising model on sufficiently ‘dense’ non-random graphs can be in an ordered state at all temperatures, see the example in [14].

which readily follows from the properties of p assumed in (2.1). Set $L_0 = 1$, $L_1 = X$, and

$$L_n = \sum_{j=1}^{L_{n-1}} X_{n,j}, \quad (2.3)$$

where all $X_{n,j}$ are independent copies of X , see Chapters II and III in [6] or Chapter 5 in [7]. Then L_n is the number of individuals in generation n of a Galton-Watson branching process. The process starts with one individual, and each individual in generation n produces offsprings at random, independently and with the same distribution as X . The individuals existing in all generations form the vertex set \mathbf{V} of the corresponding (genealogic) Galton-Watson graph \mathbf{G} , with undirected edges $\{x, y\} \in \mathbf{E}$ connecting each individual and its offsprings. Under the assumptions made in (2.1), the corresponding process does not extinct and $L_n \rightarrow +\infty$ with probability one. Hence, \mathbf{G} is an infinite tree – an acyclic connected graph, which has no leaves, except possibly for the root o , the degree of which coincides with the number of the corresponding offsprings L_0 . The degree of any other vertex equals the number of its offsprings plus one. By construction, the vertex degrees are independent identically distributed random variables such that

$$\text{Prob}\{k(o) = k\} = p_{k+1}, \quad \text{Prob}\{k(x) = k\} = p_k, \quad x \in \mathbf{V} \setminus \{o\}. \quad (2.4)$$

Now let us fix the graph-theoretical and probabilistic terminology used in this article. For a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ and vertices $x, y \in \mathbf{V}$, a *path* $\vartheta(x, y)$ is a sequence $\{x_0, x_1, \dots, x_n\}$, $n \in \mathbb{N}$, such that: (a) $x_0 = x$, $x_n = y$; (b) $\{x_l, x_{l+1}\} \in \mathbf{E}$; (c) neither of x_0, \dots, x_{n-1} can be repeated. The length of the path is $\|\vartheta(x, y)\| = n$; x is its origin and y is terminus. A *ray* with origin x_0 is an infinite sequence of distinct vertices $\{x_0, x_1, \dots, x_n, \dots\}$ with property (b). A graph ensemble \mathcal{G} is the collection of all possible realizations, e.g., as just described. A random graph model is a triple $(\mathcal{G}, \mathcal{F}, P)$, where \mathcal{F} is a certain family of subsets of \mathcal{G} and $P : \mathcal{F} \rightarrow [0, 1]$ is probability. For the Galton-Watson tree, P is defined by the collection $p = \{p_k\}$. For $\mathcal{A} \in \mathcal{F}$, $P(\mathcal{A})$ is the probability of the event: “the graph chosen at random from \mathcal{G} belongs to \mathcal{A} ”. Some events can occur with probability either zero or one – the celebrated zero-one law. In the Galton-Watson tree, if \mathcal{A} is the set of all realizations with $k(o) = 1$, then $P(\mathcal{A}) = p_1$. We say that a property of a random graph holds with *probability one* if this property is possessed by all of the elements of some \mathcal{A} such that $P(\mathcal{A}) = 1$. Note that this \mathcal{A} need not be the whole ensemble \mathcal{G} . Usually, when one deals with models with independent vertex degrees, the triple $(\mathcal{G}, \mathcal{F}, P)$ does not appear explicitly. This will also be the case in the rest of this paper.

2.2 The configuration model

Let $p := \{p_k\}_{k \geq 2}$ be as in (2.1), and

$$\hat{p}_k := kp_k/a, \quad k \geq 2. \quad (2.5)$$

“Uncorrelated complex networks“ were ‘defined’ in [3, 12, 13] by means of the following conditions. The main one is that “the neighbor of a vertex has degree k with probability \hat{p}_k ”, which can be interpreted as the condition that in every path $\{x_0, x_1, \dots, x_n\}$, the probability that $k(x_i)$ takes value k is \hat{p}_k . For convenience, we shall call it a *path property*. Another important assumptions are: (a) all $k(x)$ are “uncorrelated” (in fact, independent); (b) two paths $\{x, x_1, \dots, y\}$ and $\{x, x'_1, \dots, z\}$, with $x_1 \neq x'_1$ and $y \neq z$, never intersect each other, i.e., the graph is a tree². These assumptions are consistent with the following mathematical models. The first one is the ordinary Galton-Watson tree with p replaced by \hat{p} , in [10] it is $T(\rho, \infty)$, see page 567 thereof. A slightly different version is the so called *configuration model*, c.f. page 1279 in [3]. This model is a rooted tree with independent vertex degrees such that

$$\text{Prob}(k(o) = k) = p_k, \quad \text{Prob}(k(x) = k) = \hat{p}_k. \quad (2.6)$$

This model³ was studied in [10, 11], in [10] it appears as $T(P, \rho, \infty)$. The main characteristic feature of both $T(P, \rho, \infty)$ and $T(\rho, \infty)$ is that *each* path $\vartheta(x, y)$, $x \neq o$, has the path property described above. The inconsistency with the root degrees does not change the global behavior of the graph, and hence of the corresponding model (1.1).

The construction of the configuration model can be visualized as follows. Consider a countable set of ‘points’ (a configuration), each of which is given a random mark⁴ $k \in \mathbb{N}$ with probability distribution p as in (2.1). The marks of different points are independent. Then the graph is obtained as follows. The root degree is drawn from \mathbb{N} according to $\text{Prob}(k(o) = k) = p_k$. If the result is m , the root receives m neighbors – ‘particles’ independently drawn from the configuration with probability proportional to the particle mark, i.e., Ck . Then the (unconditional) probability of drawing some particle, which bears mark k , is Ckp_k with $C = 1/a$ since the total probability should sum up to one. In the result, each root neighbor has degree k with probability \hat{p}_k . In the same way one picks the next-neighbors of o keeping in mind that the only common element of the neighborhoods of different vertices is the root itself. This procedure is continued ad infinitum until the whole tree is constructed. An important observation here is that, except for the root degree, the distribution of $k(x)$ is given by \hat{p} , whereas p itself is the distribution of marks – auxiliary objects used in the graph construction.

²In [3, 12], the graphs were called “tree-like” if the intersections are negligible. Mathematical definitions of tree-like graphs can be found in [10, 11, 15].

³Also models which *locally converge* to $T(P, \rho, \infty)$, cf. Definition 2.1 in [10].

⁴According to [3], a mark is the number of stubs.

2.3 The proposed model

Here the main object of our study in this article is introduced. This is a random tree model, also consistent with the assumptions of [3, 12, 13]. As in the case of the Galton-Watson tree, we begin with the description of the corresponding branching process. Let p , \hat{p} , and a be as in (2.1) and (2.5). An additional parameter is $s \in \mathbb{N}$. The whole population falls into two types – ‘distinguished’ and ‘ordinary’ individuals. At the beginning, ordinary individuals are absent and there is one distinguished individual, which independently produces m offsprings with probability \hat{p}_m . If $m \leq s$, all of them are set to be distinguished. For $m > s$, s offsprings are set to be distinguished and the remaining $m - s$ ones are set to be ordinary. Each ordinary individual produces independently k ordinary individuals with probability p_{k+1} . Each distinguished individual, other than the initial one, produces independently m offsprings with probability \hat{p}_{m+1} . As above, if $m \leq s$, then all of the offsprings are distinguished. For $m > s$, s offsprings are distinguished and the remaining $m - s$ ones are ordinary. This process is repeated ad infinitum. For $n \in \mathbb{N}_0$, by \hat{L}_n and L_n we denote the number of distinguished and ordinary individuals in generation n , respectively. Let also \hat{X} and X be independent and such that $\text{Prob}(\hat{X} = m) = \hat{p}_{m+1}$ and $\text{Prob}(X = m) = p_{m+1}$, $m \in \mathbb{N}$. Then we have, cf. (2.3),

$$\begin{aligned}\hat{L}_n &= \sum_{j=1}^{\hat{L}_{n-1}} \left[\hat{X}_{n,j} - \left(\hat{X}_{n,j} - s \right)_+ \right], \\ L_n &= \sum_{j=1}^{L_{n-1}} X_{n,j} + \sum_{j=1}^{\hat{L}_{n-1}} \left(\hat{X}_{n,j} - s \right)_+, \end{aligned} \quad (2.7)$$

where, for an integer \varkappa , we write $\varkappa_+ = \max\{0, \varkappa\}$, and $\hat{X}_{n,j}$ and $X_{n,j}$ are independent copies of \hat{X} and X , respectively. From the point of view of ordinary particles, those represented by the second term in the expression for L_n are *immigrants*, and their number in generation n is

$$Y_n = \sum_{j=1}^{\hat{L}_{n-1}} \left(\hat{X}_{n,j} - s \right)_+. \quad (2.8)$$

Then the total number of individuals in generation n can be written in the form

$$\tilde{L}_n := \hat{L}_n + L_n = \sum_{j=1}^{\hat{L}_{n-1}} \hat{X}_{n,j} + \sum_{j=1}^{L_{n-1}} X_{n,j}. \quad (2.9)$$

Since each distinguished individual can have at most s distinguished offsprings, we have that

$$\hat{L}_n \leq s^n. \quad (2.10)$$

The graph in question is the genealogic tree of the process just described. Its construction can be visualized as follows. The graph is a rooted tree with $\text{Prob}(k(o) = k) = \hat{p}_{k+1}$. Let $k(o) = m \leq s$. Then we draw m neighbors of o as in the configuration model. Denote them x_1, \dots, x_m ; by construction $\text{Prob}(k(x_l) = k) = \hat{p}_k$ for all $l = 1, \dots, m$. If $m > s$, we draw s neighbors x_1, \dots, x_s of o as in the configuration model. The remaining neighbors y_{s+1}, \dots, y_m are set to be the roots of independent Galton-Watson trees with probability distribution p , which in particular means that $\text{Prob}(k(y_l) = k) = p_k$, $l = s+1, \dots, m$. This procedure is continued ad infinitum. For $s > 1$, the obtained tree contains a subtree, comprised by an infinite number of *distinguished* rays $\{o, x_1, x_2, \dots, x_n, \dots\}$ with the mentioned path property “the neighbor of x_{n-1} has degree k with probability \hat{p}_k ”. The set of vertices of this subtree is contained in that of a Cayley tree with root o and branching number s . For $s = 1$, our graph is a size-biased Galton-Watson tree, see [7, pp. 407–412] or [16], in which there is only one distinguished ray.

In the remaining part of the article we use the following nomenclature. By $\mathbf{GW}(s, p)$, $s \in \mathbb{N}_0$, we denote the model constructed in this subsection. Then $\mathbf{GW}(1, p)$ and $\mathbf{GW}(0, p)$ are the size-biased and the ordinary Galton-Watson trees, respectively.

3 The thermodynamic states of the Ising model

3.1 The thermodynamic states

Let $G = (V, E)$ be a general tree with root o . Given $x \in V$, the distance $\rho(o, x)$ is the length of the path $\vartheta(o, x)$. For $n \in \mathbb{N}_0$, by S_n we denote the collection of vertices x such that $\rho(o, x) = n$. Put

$$V_n = \bigcup_{m=0}^n S_m,$$

and let E_n be the set of all edges with both ends in V_n . Let also E_n^b be the set of $\{x, y\}$ such that $x \in S_n$ and $y \in S_{n+1}$.

By σ_n we denote the configuration of spins in V_n , that is, $\sigma_n = \{\sigma_x : x \in V_n\}$. In the sequel, we suppose that

$$J := \sup_{\{x, y\} \in E} |J_{xy}| < \infty. \quad (3.1)$$

The Gibbs probability distribution of configurations σ_n at temperature T and $h = 0$ is the following probability measure

$$\pi_n(\sigma_n | \xi) = \frac{1}{Z_n(\xi)} \exp \left(\sum_{\{x, y\} \in E_n} (K_{xy} \sigma_x \sigma_y + K) + \sum_{\{x, y\} \in E_n^b} K_{xy} \sigma_x \xi_y \right), \quad (3.2)$$

where

$$K_{xy} := \beta J_{xy}, \quad K := \beta J, \quad \beta := 1/k_B T. \quad (3.3)$$

Note that K_{xy} can have arbitrary signs, whereas K is positive. In (3.2), the first (resp. second) summand in $\exp(\cdots)$ corresponds to the interaction of the spins in V_n with each other (resp. with the spins ξ_y fixed outside V_n). One observes that the latter interaction involves only ξ_y with $y \in S_{n+1}$, which constitute the outer boundary of V_n . For technical reasons, we add positive constants to the spin-spin interaction along each edge in E_n , c.f. (1.1). The partition function is then

$$Z_n(\xi) = \sum_{\sigma_n} \exp \left(\sum_{\{x,y\} \in E_n} (K_{xy} \sigma_x \sigma_y + K) + \sum_{\{x,y\} \in E_n^b} K_{xy} \sigma_x \xi_y \right), \quad (3.4)$$

where the summation is taken over all $\sigma_x = \pm 1$, $x \in V_n$.

Given $z \in V$, let $n_z \in \mathbb{N}_0$ be such that $z \in S_{n_z}$, i.e., $n_z = \rho(o, z)$. For these z , n_z , and for $n > n_z$,

$$M_{n,z}(\xi) = \sum_{\sigma_n} \sigma_z \pi_n(\sigma_n | \xi) = \sum_{\sigma_z = \pm 1} \sigma_z \varrho_{n,z}(\sigma_z | \xi) = \varrho_{n,z}(1 | \xi) - \varrho_{n,z}(-1 | \xi) \quad (3.5)$$

is the magnetization at $z \in V_n$ in the state $\pi_n(\cdot | \xi)$ (3.2). Here, for $\sigma_z = \alpha$,

$$\varrho_{n,z}(\alpha | \xi) = \sum_{\sigma_n: \sigma_z = \alpha} \pi_n(\sigma_n | \xi), \quad \alpha = \pm 1, \quad (3.6)$$

where the summation is taken over all σ_n with the fixed $\sigma_z = \alpha$. According to the theory of Gibbs states [1], the global thermodynamic state is unique if, for all z , the limits

$$\varrho_z(\alpha) := \lim_{n \rightarrow +\infty} \varrho_{n,z}(\alpha | \xi) \quad (3.7)$$

exist and are independent of ξ . In this case, by (3.5) the global magnetization M_z exists and

$$M_z = \lim_{n \rightarrow +\infty} M_{n,z}(\xi) = \varrho_z(1) - \varrho_z(-1).$$

As $h = 0$, we have

$$M_{n,z}(\xi) = -M_{n,z}(-\xi), \quad (3.8)$$

and hence $M_z = -M_z$. Therefore, the uniqueness of thermodynamic states occurs if and only if, for all $z \in V$ and ξ ,

$$M_z = \lim_{n \rightarrow +\infty} M_{n,z}(\xi) = 0. \quad (3.9)$$

In this case, we say that the model is in a *paramagnetic state*.

If the underlying tree G is random, then the measure (3.2) is also random, and hence (3.9) is a random event obeying the zero-one law. If the corresponding probability is one, we say that the model (1.1) on G is in a paramagnetic state with probability one.

3.2 The main statement

Let p and \hat{p} be as in (2.1) and (2.5). For $\alpha \in (0, 1)$, we set

$$b_\alpha = \sum_{k \geq s+1} (k-s)^\alpha \hat{p}_k, \quad b = \sum_{k \geq 2} k p_k \ln k, \quad (3.10)$$

and also

$$K_c = \frac{1}{4} \ln \frac{a}{a-1}, \quad \widehat{K}_c = \frac{1}{4} \ln \frac{s^\gamma + 1}{s^\gamma}, \quad \gamma = 1/\alpha. \quad (3.11)$$

In the statement below we describe the Ising model (1.1) on $\mathbf{GW}(s, p)$ graphs with p obeying (2.1).

Theorem 3.1 *For $s = 0$, the Ising model is in a paramagnetic state whenever $K < K_c$. For $s = 1$, the same holds under the condition $b < \infty$. For $s \geq 2$, the Ising model is in a paramagnetic state whenever $b_\alpha < \infty$ for some $\alpha \in (0, 1)$ and $K < \min\{K_c; \widehat{K}_c\}$.*

In Section 5, we discuss the above statement in detail. Its proof is based on two lemmas proven in the next section. In the first lemma, we describe the model (1.1) on a general tree, for which the corresponding quantities were introduced in Subsection 3.1.

Lemma 3.2 *Let $K > 0$ be such that*

$$q(K) := \exp(4K) - 1 < 1. \quad (3.12)$$

Then for each $z \in \mathbf{V}$, any $n > n_z$, and arbitrary ξ and η , we have that

$$|M_{n,z}(\xi) - M_{n,z}(\eta)| \leq 2[q(K)]^{n-n_z} |\mathbf{S}_n|. \quad (3.13)$$

In the next lemma, we describe the model $\mathbf{GW}(s, p)$ introduced in Subsection 2.3.

Lemma 3.3 *Let L_n be as in (2.7), $s \geq 2$, and $b_\alpha < \infty$ for some $\alpha \in (0, 1)$. Then, for every c obeying $c > s^\gamma$ and $c \geq a - 1$, with probability one*

$$c^{-n} L_n \rightarrow W \in [0, \infty), \quad \text{as } n \rightarrow \infty.$$

Proof of Theorem 3.1: For $\underline{s = 0}$, by construction $|\mathbf{S}_n| = L_n$ is the number of individuals in generation n , see (2.3), and $a - 1$ is the mean number of offsprings in the corresponding Galton-Watson process. Then the limit

$$\lim_{n \rightarrow +\infty} \frac{L_n}{(a-1)^n} \quad (3.14)$$

with probability one exists and is finite, see (2.2) and Proposition 1.3 in [6, page 20]. If $K < K_c$, then $q(K) < q(K_c) = 1/(a-1) < 1$, and hence we can apply (3.13), which yields that with probability one

$$[q(K)(a-1)]^n \frac{L_n}{(a-1)^n} \rightarrow 0, \quad n \rightarrow +\infty, \quad (3.15)$$

which in turn by (3.13) yields (3.9), and hence the proof for this case.

For $s = 1$, by (2.10) we have $\tilde{L}_n = 1$, and hence $|\mathbf{S}_n| = \tilde{L}_n = 1 + L_n$. Since $b < \infty$, with probability one we have

$$\sum_{n \geq 1} (a-1)^{-n} Y_n < \infty, \quad (3.16)$$

see Proposition 6.2 in [6, page 50]. Then by Theorem 6.1 in [6, page 50], $\tilde{L}_n/(a-1)^n$ with probability one tends to a finite limit. Then the proof follows as in the case of $s = 0$.

For $s \geq 2$, the random variables Y_n with different n are no more identically distributed, and hence $b < \infty$ is not enough to get (3.16). Instead we use a more restrictive condition $b_\alpha < \infty$, under which we get, see Lemma 3.3, that $c^{-n}|\mathbf{S}_n| = c^{-n}\tilde{L}_n = c^{-n}\hat{L}_n + c^{-n}L_n$ with probability one tends to a finite limit since $c > s$, cf. (2.10). Note that

$$K(c) := \frac{1}{4} \ln \frac{c+1}{c} \leq \min\{K_c; \hat{K}_c\}.$$

Hence, for $K < K(c)$, we have $q(K) < 1/c$, and the proof follows as in (3.15).

4 The proof of the lemmas

4.1 The proof of Lemma 3.2

Let $G = (V, E)$ be a general tree. For a path ϑ , by E_ϑ we denote the set of edges $\{x_{l-1}, x_l\}$ of ϑ . Since G is a tree, there exists exactly one path $\vartheta(x, y)$ for any x and y . Recall that E_n stands for the set of edges with both ends in V_n . Thus, $G_n = (V_n, E_n)$ is a finite graph; $G' = (V', E')$ is called a subgraph of G_n if $V' \subset V_n$ and $E' \subset E_n$. A path ϑ is said to be in G' if $E_\vartheta \subset E'$. Two vertices $z, x \in V'$ are said to be *disconnected* in G' if there is no path $\vartheta(z, x)$ in G' .

By (3.5) and (3.2), employing replica spins $\tilde{\sigma}$ we obtain the following

Meyer-like expansion

$$\begin{aligned}
M_{n,z}(\xi) - M_{n,z}(\eta) &= \sum_{\sigma_n, \tilde{\sigma}_n} (\sigma_z - \tilde{\sigma}_z) \pi_n(\sigma_n|\xi) \pi_n(\tilde{\sigma}_n|\eta) \\
&= \frac{1}{Z_n(\xi)Z_n(\eta)} \sum_{\sigma_n, \tilde{\sigma}_n} (\sigma_z - \tilde{\sigma}_z) \prod_{\{x,y\} \in E_n} (1 + \Gamma_{xy}) \Psi_n(\xi, \eta) \\
&= \frac{1}{Z_n(\xi)Z_n(\eta)} \sum_{E' \subset E_n} \sum_{\sigma_n, \tilde{\sigma}_n} (\sigma_z - \tilde{\sigma}_z) \Gamma(E') \Psi_n(\xi, \eta),
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
\Gamma(E') &= \prod_{\{x,y\} \in E'} \Gamma_{xy}, \\
\Gamma_{xy} &= \exp[(K_{xy}\sigma_x\sigma_y + K) + (K_{xy}\tilde{\sigma}_x\tilde{\sigma}_y + K)] - 1, \\
\Psi_n(\xi, \eta) &= \prod_{\{x,y\} \in E_n^b} \exp[K_{xy}(\sigma_x\xi_y + \tilde{\sigma}_x\eta_y)].
\end{aligned} \tag{4.2}$$

For simplicity, we do not indicate the dependence of the latter functions on the spins. One observes that each $\Gamma_{xy} \geq 0$ due to our choice of the spin-spin interactions in (3.2), cf. (3.1), and that $\Psi_n(\xi, \eta)$ depends only on σ_x and $\tilde{\sigma}_x$ with $x \in S_n$. Fix some E' in the last line in (4.1) and consider the subgraph G' with the edge set E' and the vertex set V_n . If in G' the vertex z is disconnected from each $x \in S_n$, then the sums over the spins $\sigma_z, \tilde{\sigma}_z$ and over $\sigma_x, \tilde{\sigma}_x$ with $x \in S_n$ get independent and hence the left-hand side of (4.1) vanishes as the term $(\sigma_z - \tilde{\sigma}_z)$ is antisymmetric with respect to the interchange $\sigma \leftrightarrow \tilde{\sigma}$, whereas all Γ_{xy} are symmetric and the only break of this symmetry is related with the fixed boundary spins $\xi(y)$ and $\eta(y)$, $y \in S_{n+1}$. Therefore, each non-vanishing term in (4.1) corresponds to a path $\vartheta(z, x)$ connecting z to some $x \in S_n$. Let us take this into account and rewrite (4.1) as the sum over the subsets of E_n containing the edges of at least one such path. Let $\Theta_n(z)$ be the set of all paths connecting z to S_n , and for $\vartheta \in \Theta_n(z)$, let \mathcal{E}_ϑ be the family of subsets of E_n each of which contains E_ϑ . That is, $\mathcal{E}_\vartheta = \{E' \subset E_n : E_\vartheta \subset E'\}$. Note that, for distinct $\vartheta, \vartheta' \in \Theta_n(z)$, the corresponding families \mathcal{E}_ϑ and $\mathcal{E}_{\vartheta'}$ are not disjoint – they include those E' which contain both E_ϑ and $E_{\vartheta'}$. Finally, set

$$\mathcal{E} = \bigcup_{\vartheta \in \Theta_n(z)} \mathcal{E}_\vartheta, \tag{4.3}$$

that is, \mathcal{E} contains all sets of edges $E' \subset E_n$ such that the corresponding graph G' contains at least one path connecting z to some $x \in S_n$. Then (4.1) takes the form

$$M_{n,z}(\xi) - M_{n,z}(\eta) = \frac{1}{Z_n(\xi)Z_n(\eta)} \sum_{E' \in \mathcal{E}} \sum_{\sigma_n, \tilde{\sigma}_n} (\sigma_z - \tilde{\sigma}_z) \Gamma(E') \Psi_n(\xi, \eta). \tag{4.4}$$

In view of the positivity of all $\Gamma_{xy} \geq 0$, this yields

$$\begin{aligned} |M_{n,z}(\xi) - M_{n,z}(\eta)| &\leq \frac{2}{Z_n(\xi)Z_n(\eta)} \sum_{E' \in \mathcal{E}} \sum_{\sigma_n, \tilde{\sigma}_n} \Gamma(E') \Psi_n(\xi, \eta) \quad (4.5) \\ &\leq \frac{2}{Z_n(\xi)Z_n(\eta)} \sum_{\sigma_n, \tilde{\sigma}_n} \sum_{\vartheta \in \Theta_n(z)} \Gamma(E_\vartheta) \sum_{E' \in \mathcal{E}_\vartheta} \Gamma(E' \setminus E_\vartheta) \Psi_n(\xi, \eta). \end{aligned}$$

Note that the sum $\sum_{\vartheta \in \Theta_n(z)} \sum_{E' \in \mathcal{E}_\vartheta}$ contains the same summands as $\sum_{E' \in \mathcal{E}}$, see (4.3), but a part of them are repeated, which yields the second \leq in (4.5). From (4.2) and (3.12) we see that $0 \leq \Gamma_{xy} \leq q(K)$, for any $x, y \in V$. Hence,

$$\Gamma(E_\vartheta) \leq [q(K)]^{\|\vartheta\|},$$

which yields in (4.5)

$$\begin{aligned} |M_{n,z}(\xi) - M_{n,z}(\eta)| &\leq \frac{2}{Z_n(\xi)Z_n(\eta)} \sum_{\vartheta \in \Theta_n(z)} [q(K)]^{\|\vartheta\|} \quad (4.6) \\ &\times \sum_{\sigma_n, \tilde{\sigma}_n} \prod_{\{x,y\} \in E_\vartheta} (1 + \Gamma_{xy}) \sum_{E'' \subset E_n \setminus E_\vartheta} \Gamma(E'') \Psi_n(\xi, \eta) \\ &= 2 \sum_{\vartheta \in \Theta_n(z)} [q(K)]^{\|\vartheta\|}. \end{aligned}$$

Here we have taken into account that, see (4.1),

$$\begin{aligned} &\sum_{\sigma_n, \tilde{\sigma}_n} \prod_{\{x,y\} \in E_\vartheta} (1 + \Gamma_{xy}) \sum_{E'' \subset E_n \setminus E_\vartheta} \Gamma(E'') \Psi_n(\xi, \eta) \\ &= \sum_{\sigma_n, \tilde{\sigma}_n} \prod_{\{x,y\} \in E_n} (1 + \Gamma_{xy}) \Psi_n(\xi, \eta) = Z_n(\xi)Z_n(\eta). \end{aligned}$$

Note that a similar construction was used in deriving (2.16) and (2.17) in [17], see also Lemma 3.1 in [14]. Since n_z is defined by the condition $z \in S_{n_z}$, the shortest path in $\Theta_n(z)$ has length $n - n_z$. Furthermore, for every $x \in S_n$, there exists exactly one path $\vartheta \in \Theta_n(z)$. Then, for $q(K) \leq 1$, the estimate (3.13) follows from (4.6).

4.2 The proof of Lemma 3.3

If we prove that, for $c > s^\gamma$, with probability one

$$\sum_{n \geq 1} c^{-n} Y_n < \infty, \quad (4.7)$$

see (2.8), then the proof that $L_n/c^n \rightarrow W \in [0, +\infty)$, with probability one, follows as in the proof of Theorem 6.1 in [6, page 50]. The first summand of L_n in (2.7) is under control in view of the assumed property $c \geq a - 1$.

As $\alpha \in (0, 1)$, by the standard Minkowski inequality, cf. [18, Theorem 8, page 319], as well as by (2.10) and (3.10), we have

$$\langle Y_n^\alpha \rangle \leq \sum_{j=1}^{s^{n-1}} \left\langle \left(\widehat{X}_j - s \right)_+^\alpha \right\rangle = s^{n-1} b_\alpha.$$

Then, for any $q > c^\gamma$, by Markov's inequality, cf. [18, page 311],

$$P_n := \text{Prob}(q^{-n} Y_n > 1) = \text{Prob}(Y_n^\alpha > q^{\alpha n}) \leq \frac{b_\alpha}{s} \left(\frac{s}{q^\alpha} \right)^n. \quad (4.8)$$

On the other hand, for $c > s^\gamma$, we take $q \in (s^\gamma, c)$ and rewrite

$$\sum_{n \geq 1} c^{-n} Y_n = \sum_{n \geq 1} \left(\frac{q}{c} \right)^n (q^{-n} Y_n). \quad (4.9)$$

For such q , by (4.8) we have that $\sum_{n \geq 1} P_n < \infty$. Thus, the Borel-Cantelli lemma, cf. [18, page 320], yields that with probability one only finitely many of the events $q^{-n} Y_n > 1$ may occur, which by (4.9) readily yields (4.7).

5 Concluding comments and remarks

5.1 The graph model

Let us look at the structure of generation n in $\mathbf{GW}(s, p)$, cf. (2.7) – (2.10). For $s = 1$, we deal with a size-biased Galton-Watson tree, cf. [7] and [16], and $b < \infty$ is known as the $X \log X$ condition of the Kesten-Stigum theorem, see also [6, page 23]. If it holds, the influence of the distinguished ray on the structure of \mathbf{S}_n is asymptotically negligible, see Section 3 in [16]. For $s \geq 2$, the number of distinguished vertices \widehat{L}_n in generation n increases in such a way that the sequence $\{\sigma^{-n} \widehat{L}_n\}_{n \in \mathbb{N}}$ with probability one tends to a random variable \widehat{W} such that $\text{Prob}(\widehat{W} > 0) = 1$ and $\langle \widehat{W} \rangle = 1$. Here

$$\sigma = s - (s - 1)\hat{p}_2 - (s - 2)\hat{p}_3 - \cdots - \hat{p}_s.$$

Note that $\sigma \in (1, s]$, and $\sigma = s$ if and only if $p_2 = \cdots = p_s = 0$, and hence the least value of X is s . In this case, $a - 1 > s$. If $a - 1 > s^\gamma$, L_n asymptotically ‘behaves’ like in the ordinary Galton-Watson tree, which means that the production of offsprings with probability p ‘dominates’ the immigration from the distinguished part of the population, which is always the case for $\mathbf{GW}(1, p)$, cf. (2.2). However, for $a - 1 \leq s^\gamma$, the main contribution into L_n comes from the immigration. If $a - 1$ is close to one and $s^\gamma \gg 1$, even

for big n the structure of (V_n, E_n) is quite close to that of the corresponding subgraph of $\mathbf{GW}(0, \hat{p})$, i.e., as in the case of the configuration model. Note that s^γ is big whenever (a) s is big, and hence the distinguished subgraph is ‘big’; (b) α is small, and hence the offspring production in the distinguished subgraph is very intensive.

Noteworthy, the procedure of formation of edges in $\mathbf{GW}(s, p)$ resembles that of [8, 9]. The ‘value’ specified in [9] by a positive ω , in our case appears as a mark k , distributed according to p with p_k ‘decreasing at infinity’, cf. “inverse mass-action principle” of [9]. Distinguished individuals have the right to choose neighbors. Each chooses according to its own mark at most s distinguished neighbors, preferring those with big marks. The restriction of the number of distinguished neighbors to s corresponds to the restriction of the number of outgoing edges in the Cameo principle [9].

One observes that $\mathbf{GW}(s, p)$ is a natural generalization of the size-biased Galton-Watson tree $\mathbf{GW}(1, p)$. In a separate work, we plan to study this model in more detail. Another generalization of $\mathbf{GW}(1, p)$ can be the model in which the distinguished part of \hat{X} is not $\hat{X} - (\hat{X} - s)_+$ like in $\mathbf{GW}(s, p)$, cf. (2.7), but some more general increasing function of \hat{X} , or is random. Further generalizations of this kind can be obtained in the approach of [19].

5.2 The thermodynamic states

In 1989, R. Lyons [15] proved that, for any tree, the critical temperature of the model (1.1) with $h = 0$ and $J_{xy} = J > 0$ is

$$T_c = \frac{J}{k_B \coth^{-1} \varrho}, \quad (5.1)$$

where ϱ is the *branching number* of the underlying tree. For an infinite (almost surely non-extinct) Galton-Watson tree, with probability one ϱ equals the mean number of offsprings, see Proposition 6.4 in [20]. Hence, for $\mathbf{GW}(0, \hat{p})$, we have $\varrho = \langle k^2 \rangle / \langle k \rangle - 1$, which yields in (5.1)

$$T_c = 2J / k_B \ln \frac{\langle k^2 \rangle}{\langle k^2 \rangle - 2\langle k \rangle}. \quad (5.2)$$

The very same expression for T_c was ‘rediscovered’ in 2002 in [12, 13]. It is significant that Lyon’s paper [15] was quoted⁵ in [3] and even the same formula (5.1) was discussed, see the text between equations (89) and (90) on page 1304 in [3] or the text between equations (89) and (90) on page 33 in arXiv:0705.0010.

Now let us turn to the discussion of Theorem 3.1, which describes the model (1.1) with *arbitrary* signs⁶ of the intensities J_{xy} . The only condition

⁵In the preprint arXiv:0705.0010 version of [3], both papers [15, 20] were quoted.

⁶The antiferromagnetic case $J_{xy} = -J$, $J > 0$, can be reduced to the ferromagnetic one by changing signs of σ_x , $x \in S_{2n}$.

is that they are uniformly bounded, cf. (3.1). As follows from the estimate (3.13), the model (1.1) on a general tree can be in a paramagnetic state if the number of vertices in the sphere S_n admits the control $|S_n| \leq c^n$ for some $c > 1$. Note that in the example of [14], $|S_n| \sim n!$. For graphs other than trees⁷, one has to control $\Theta_n(z)$, cf. (4.6). The results of Theorem 3.1 can naturally be extended to models with continuous spins and bounded interaction, as e.g. in [14]. In a separate work, we plan to study the case of unbounded interaction intensities, which includes also random J_{xy} , as e.g. in the Edwards-Anderson model. For a ferromagnetic Ising model on a $\mathbf{GW}(0, p)$, the exact value of the critical temperature is given in (5.1) with $\varrho = a - 1$. The corresponding result of Theorem 3.1 is that the model is in a paramagnetic state for

$$T > 4J \Big/ k_B \ln \frac{a}{a-1} > T_c.$$

It is naturally less precise as we cover the case of arbitrary signs of J_{xy} .

Regarding the model $\mathbf{GW}(0, \hat{p})$ studied in [12, 13], by Theorem 1.1 we get that the Ising model with arbitrary signs of J_{xy} obeying (3.1) is in a paramagnetic state if

$$T > 4J \Big/ k_B \ln \frac{\langle k^2 \rangle}{\langle k^2 \rangle - 1}.$$

It is believed, cf. [22], that the Internet can well be modeled as a scale-free graph with $\lambda = 5/2$, which corresponds to the choice of $\alpha < 1/2$, see (3.10). As a byproduct of the result of Theorem 3.1, we obtain the following, cf. [22, 23],

Proposition 5.1 *Let $\theta \in (0, 1)$ be the Bernoulli bond percolation probability on $\mathbf{GW}(s, p)$ graphs with $s \geq 2$ and p obeying (2.1) and $b_\alpha < \infty$. Then with probability one there is no giant component if*

$$\theta < \min\{q(K_c); q(\hat{K}_c)\},$$

where $q(K)$ is given by (3.12), and K_c and \hat{K}_c are as in Theorem 1.1.

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⁷For more on phase transitions on graphs see [21].

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